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The Calculation of Eigenvalues for the Stationary Perturbation of Poiseuille Flow Using Initial Value Methods

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1. INTRODUCTION

Bramley and Dennis [1] obtained the eigenvalues of a stationary perturbation of Poiseuille flow for the viscous flow in a straight channel and solved the problem by a spectral method using Chebyshev series. The present paper is an extension and solves the same problem using initial-value methods: Riccati transformation, the compound matrix method, and orthogonalisation. It is possible to calculate additional complex eigenvalues using initial value methods and these eigenvalues are presented. The spectral method gives the eigenvalues without searching on a real line or over a complex plane. The methods used in this paper require an initial approximation to be given and will then give the accurate value for the eigenvalue. In the case of complex eigenvalues, it is not always easy to find an initial approximation good enough for the method to work. In the previous paper eigenvalues were obtained for various Reynolds numbers R and with the spectral method you choose a particular R and then all the eigenvalues (both complex and real) are calculated. In this paper for a particular R only one eigenvalue α is calculated at a time. The initial value methods require much less computer time. It is advantageous to have a method which can check a single result. In particular the methods used in this paper are useful to check for spurious eigenvalues which are sometimes generated when using Chebyshev series. The Chebyshev series method can always be used to check itself by increasing the number of Chebyshev coefficients, but the initial value methods have the distinct advantage that they require far less computer time.

The results obtained using initial value methods are in agreement with those of [1] and are not repeated with the exception that more complex

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eigenvalues for $10 \leqslant R \leqslant 2000$ are given. Some of these were not obtained using the spectral method due to numerical difficulties and the large amount of computer time required.

2. Equations

With the formulation of the problem being the same as the previous paper we will rely on that paper for a full explanation and derivations of the equations. Essentially we solve the differential eigenvalue problem,

$$\phi'^{v} + 2\alpha^2\phi'' + \alpha^4\phi + \alpha R[\tfrac{3}{2}(1-y^2)(\phi'' + \alpha^2\phi) + 3\phi] = 0 \tag{2.1}$$

with boundary conditions

$$\phi(\pm 1) = \phi'(\pm 1) = 0. \tag{2.2}$$

The eigenfunction $\phi(y)$ is a function of y, the transverse non-dimensional distance measured from the centre of the tube. The Reynolds number is R $(0 \leqslant R \leqslant 2000)$ and α is the eigenvalue.

The eigenfunction ϕ will either by odd or even and the associated eigenvalues are referred to being either odd or even.

$$\text{If } \phi \text{ is even then } \phi'(0) = \phi'''(0) = 0;$$
$$\text{if } \phi \text{ is odd then } \phi(0) = \phi''(0) = 0.$$

The problem now splits into two similar problems, and we will only give details for the even problem where the boundary conditions are

$$\phi'(0) = \phi'''(0) = 0 \qquad \text{and} \qquad \phi(1) = \phi'(1) = 0.$$

The odd problem follows through in a similar manner. The three methods are described in the next three sections. All three methods are fully described in the relevant references and so only the main details are presented in this paper.

3. Riccati Method

The Riccati method was first developed by Scott [2, 3] and subsequently expanded by many authors. We refer to the explanation given by Wilks and Bramley [4]. The method is based on the introduction of a 2×2 matrix $R(y)$ using the transformation

$$\boldsymbol{u}(y) = R(y)\,\mathbf{v}(y), \tag{3.1}$$

where $[\mathbf{u}(y)]^T = (\phi'(y), \phi'''(y))$ and $[\mathbf{v}(y)]^T = (\phi(y), \phi''(y))$.

Equation (2.1) can be rewritten in the form

$$\begin{aligned} \frac{d\mathbf{u}}{dy} &= A(y,\alpha)\,\mathbf{u} + B(y,\alpha)\,\mathbf{v} \\ -\left(\frac{d\mathbf{v}}{dy}\right) &= C(y,\alpha)\,\mathbf{u} + D(y,\alpha)\,\mathbf{v} \end{aligned} \tag{3.2}$$

subject to boundary conditions for the even eigenfunction

$$\mathbf{u}(0) = 0 \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(1) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{v}(1) = \mathbf{0},$$

where A, B, C, D are 2×2 matrices.

It is readily shown that R satisfies the matrix Riccati equation,

$$R'(y) = B(y,\alpha) + A(y,\alpha)\,R(y) + R(y)\,D(y,\alpha) + R(y)\,C(y,\alpha)\,R(y), \tag{3.3}$$

where $R = \left[\begin{smallmatrix} R_1 & R_2 \\ R_3 & R_4 \end{smallmatrix}\right]$ is a 2×2 matrix.

Let us consider the case of complex α. It can be shown that the complex eigenvalues come in complex conjugate pairs. In the complex case all the manipulations, etc., can be done in complex algebra with the one exception that Eq. (3.3) must be integrated with respect to the real variable y. The boundary condition $\mathbf{u}(0) = \mathbf{0}$ permits the integration of (3.3) using the initial condition $R(0) = 0$. The matrix R has four complex elements giving eight equations with real coefficients to integrate. Equation (3.3) is now integrated from $y = 0$ to $y = 1$, at which point we need

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(1) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{v}(1) = \mathbf{0},$$

which means that

$$\operatorname{Det}\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right] = 0.$$

This gives

$$\operatorname{Det}\begin{bmatrix} R_1 & R_2 \\ 1 & 0 \end{bmatrix} = 0,$$

which is satisfied when $R_2 = 0$.

Equation (2.1) is very much like the Orr–Sommerfeld equation and the method is described in detail by Sloan and Wilks [5]. An initial estimate of the complex eigenvalue is required and Eq. (3.3) is integrated from $y = 0$ to

$y = 1$ with three approximate values of α and then a linear combination of the three solutions is formed to give $R_2 = 0$.

The case where α is real is more complicated and the method is explained in Wilks and Bramley [4] and Sloan [6]. There are many ways to use the Riccati method but the way found to be best for this particular problem is to start from $y = 1$. At $y = 1$ put $\mathbf{U}(y) = E(y)\,\mathbf{V}(y)$, where $\mathbf{U}(y) = [\phi(y), \phi'(y)]^T$ and $\mathbf{V}(y) = [\phi''(y), \phi'''(y)]^T$. We now start at $y = 1$ with $E(y) = 0$ and integrate a little way towards $y = 0$ and then switch to the R-system method explained in [4, 6]. The matrix equation for E will be similar to Eq. (3.3). If any of the elements of R become large we have to introduce an S-system, where

$$\mathbf{v}(y) = S\mathbf{u}(y) \tag{3.4}$$

and $S = R^{-1}$. The method of switching between the R-system and the S-system is explained in the references. At $x = 0$ on the R system we require $\mathbf{u} = \mathbf{0}$ and so $\det R = 0$. In practice we choose a real value of α and integrate to $y = 0$ and note the value of $\det R$. We vary α until we have $\det R$ of an opposite sign and then use the rule of false position to calculate the eigenvalue α, which gives $\det R = 0$.

4. Compound Matrix Method

In this method and the method of orthogonalisation of Section 5 there is no difference in the explanation of the theory between the cases for complex α and real α. The compound matrix method was developed by Gilbert and Backus [7] and resurrected by Ng and Read [8]. Define the vector $\boldsymbol{\phi} = [\phi, \phi', \phi'', \phi''']^T$ and let $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ be two linear independent solutions satisfying the even case boundary conditions $\phi'(0) = \phi'''(0) = 0$. Then we have

$$\boldsymbol{\phi}_1(0) = [1, 0, 0, 0]^T \qquad \text{and} \qquad \boldsymbol{\phi}_2(0) = [0, 0, 1, 0]^T.$$

Consider the 4×2 solution matrix

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \\ \phi_1''' & \phi_2''' \end{bmatrix}. \tag{4.1}$$

Define p_i $(i = 1, 2, 3, 4, 5, 6)$ as the six minors of this matrix; then

$$\begin{aligned} p_1 &= \phi_1\phi_2' - \phi_1'\phi_2, \\ p_2 &= \phi_1\phi_2'' - \phi_1''\phi_2, \\ p_3 &= \phi_1\phi_2''' - \phi_1'''\phi_2, \\ p_4 &= \phi_1'\phi_2'' - \phi_1''\phi_2', \\ p_5 &= \phi_1'\phi_2''' - \phi_1'''\phi_2', \\ p_6 &= \phi_1''\phi_2''' - \phi_1'''\phi_2''. \end{aligned} \tag{4.2}$$

Following Ng and Reid [8] it can be shown that $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5, p_6)^T$ satisfies the system of differential equation

$$\mathbf{p}'(y) = A(y)\,\mathbf{p}(y), \tag{4.3}$$

where $A(y)$ is the 6×6 matrix given by

$$A(y) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -C_2 & 0 & 0 & C_1 & 0 & 1 \\ 0 & -C_2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} C_1 &= -2\alpha^2 - \tfrac{3}{2}R\alpha(1 - y^2) \\ C_2 &= -\alpha^4 - 3R\alpha(\tfrac{1}{2}(1 - y^2)\,\alpha^2 + 1). \end{aligned}$$

At $y = 0$ all the p_i's are zero except y_2 and we choose $p_2(0) = 1$. The system of equations is now integrated with respect to the real variable from $y = 0$ to $y = 1$. In order to satisfy the boundary conditions at $y = 1$ we must have

$$\begin{aligned} \lambda\phi_1 + \mu\phi_2 &= 0 \\ \lambda\phi_1' + \mu\phi_2' &= 0 \end{aligned} \tag{4.4}$$

for all λ and μ.

Hence the termination condition at $y = 1$ is

$$\det \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = p_1 = 0.$$

As in the previous chapter for both real and complex cases we choose an initial guess for α and then modify it until $p_1 = 0$.

5. Orthonormalisation Method

The method of orthonormalisation has been developed by Godunov [9] and Conte [10]. Following the notation of the previous section when α is real we compute two basis functions $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ but this time integrate the system of ordinary differential equations using a standard Runge–Kutta procedure. If at any stage the two basis vectors cease to be independent of each other, then we must create new bases which are independent of each other. The basis vectors usually show themselves to be dependent on each other when some of the elements become large. At this stage we generate new basis vectors which are independent of each other (orthogonal), from a linear combination of the old basis vectors, i.e., we calculate $\boldsymbol{\phi}_{1\,\text{new}}$ and $\boldsymbol{\phi}_{2\,\text{new}}$ from $\boldsymbol{\phi}_{1\,\text{old}}$ and $\boldsymbol{\phi}_{2\,\text{old}}$ so that

$$\left.\begin{aligned} (\boldsymbol{\phi}_1)_{\text{new}} \cdot (\boldsymbol{\phi}_1)_{\text{new}} &= 1, \\ (\boldsymbol{\phi}_2)_{\text{new}} \cdot (\boldsymbol{\phi}_2)_{\text{new}} &= 1, \\ (\boldsymbol{\phi}_1)_{\text{new}} \cdot (\boldsymbol{\phi}_2)_{\text{new}} &= 0. \end{aligned}\right\} \tag{5.1}$$

This re-orthonormalisation process is carried out as many times as required during the integration from $y = 0$ to $y = 1$. A record of the orthonormalisations is only kept if the eigenfunction is required (see Sloan [6]). For the boundary conditions at $y = 1$ to be satisfied

$$\begin{aligned} \mu\phi_1(1) + \lambda\phi_2(1) &= 0 \\ \mu\phi_1'(1) + \lambda\phi_2'(1) &= 0 \end{aligned} \tag{5.2}$$

for all μ and λ so at $y = 1$

$$\phi_1\phi_2' - \phi_1'\phi_2 = 0. \tag{5.3}$$

The process now continues as in the previous section for real α and α is varied until condition (5.3) is satisfied.

For complex α the linear system of ordinary differential equations is now of order 8 and we have four basis vectors; the process follows in a similar way to that of the real case.

6. Numerical Results

The numerical results agree with the published results of Bramley and Dennis [1]. Using the initial value methods described in this paper we are able to obtain accurate results for complex α with real part $\geqslant 0$. The spectral method could possibly have given these results but would have used a large amount of computer time. The number of Chebyshev polynomials used in the approximation would have needed to be increased and thus the dimension of the matrix used could have been so large that rounding errors made it impossible to get a more accurate value. In the initial value methods all that is required is a reasonably accurate starting value. The new results are given in Tables I and II. In Tables 1 and 2 of Bramley and Dennis [1] only the eigenvalues that could be calculated correct to four significant figures using the Chebyshev approximation were presented.

It will be noted that in Table I there is a complex eigenvalue for Reynolds number, $R = 50$, missing. Many attempts were made to try to obtain this eigenvalue, but to no avail. The complex eigenvalue with real part near 6 is obtained for $R = 48$ and 52 but cannot be obtained for $R = 50$ by any method. The spectral method also failed to give the corresponding complex value for $R = 50$. It is thought that the reason for this phenomenon is that the complex eigenvalue and its conjugate become two real eigenvalues. Bramley and Dennis [1] explained that the number of real eigenvalues varied. If Fig. 3 of [1] is inspected closely it can be noted that at $R = 50$ the fourth curve up just dips below the line $R = 50$, thus giving two real eigenvalues, and the values of these eigenvalues are very close to 6, approximately the value of the real part of the missing eigenvalue. It is suggested that at this point the discriminant becomes > 0 and instead of a complex eigenvalue and its conjugate there are two real eigenvalues. The eigenvalue α has to be chosen as either real or complex at the beginning of the calculation and the programs for the two cases are different. This being so, when we search for the missing complex eigenvalue the program fails to converge but if we search for a real eigenvalue it is possible to find two, which are very near to the value of the real part of the missing complex eigenvalues. This is not the only point where this phenomenon could show itself but it is the only case in the values of R which are tabulated in Tables I and II.

Many problems arise when using the methods discussed in this paper and a few are stated below. Finding an initial approximation for α can be a problem when α is complex because α may be anywhere on the complex plane and therefore easily missed. With complex α it is possible to use the spectral method to either calculate α or at least obtain a good enough initial approximation for use with the initial value methods. The Riccati method may cause problems in deciding where to switch from one system to another. Some of the elements of the matrices can rapidly become very large. This

TABLE I

Complex Odd Eigenvalues with Positive Real Part for $10 \leqslant R \leqslant 2000$

R	Eigenvalues		
10	$3.37037 + i0.51693$	$6.53508 + i1.08843$	$9.63135 + i1.29634$
25	$2.66742 + i0.25719$	$6.24730 + i0.85814$	$9.18932 + i0.73898$
50	$2.69422 + i0.21555$		$9.44649 + i0.81230$
100	$2.82446 + i0.40413$	$6.38885 + i0.34776$	$9.67935 + i0.46076$
250	$2.78306 + i0.27633$	$6.17846 + i0.37240$	$9.44428 + i0.45332$
500	$2.71137 + i0.25550$	$6.15742 + i0.27204$	$9.38800 + i0.35151$
1000	$2.70753 + i0.15583$	$6.12043 + i0.20051$	$9.31773 + i0.16143$
2000	$2.66618 + i0.13674$	$6.06950 + i0.12724$	$9.33590 + i0.17868$

can be overcome by testing the elements very often although the method does become inefficient. There are many Riccati schemes to choose from and it is not always clear which particular scheme will give the results with the least amount of computing effort. At the higher Reynolds numbers of 1000 and 2000 each method becomes difficult to use. The Riccati method gives switching problems, the elements of the compound matrix method become too large, and the orthonormalisation method requires re-orthonormalisations at very many points between $y = 0$ and $y = 1$ so as to make the method very slow.

TABLE II

Complex Even Eigenvalues with Positive Real Part for $10 \leqslant R \leqslant 2000$

R	Eigenvalues		
10	$1.01976 + i0.77058$	$4.97904 + i0.90975$	$8.08350 + i1.20716$
25	$1.04158 + i0.60928$	$4.78856 + i0.72461$	$7.70885 + i0.85313$
50	$0.87801 + i0.51910$	$4.52791 + i0.59662$	$8.04106 + i0.70542$
100	$0.81781 + i0.45772$	$4.57519 + i0.46328$	$7.74136 + i0.55653$
250	$0.72120 + i0.37930$	$4.56894 + i0.21827$	$7.79156 + i0.41948$
500	$0.65637 + i0.33642$	$4.49069 + i0.23554$	$7.68946 + i0.17741$
1000	$0.59650 + i0.29956$	$4.45422 + i0.16968$	$7.68314 + i0.22734$
2000	$0.54194 + i0.26802$	$4.40321 + i0.15411$	$7.67001 + i0.17246$

7. Conclusion

The methods described in this paper have proved to be very useful. They are used to check the results obtained using spectral methods given in [1]. The initial value methods are more amenable than the spectral method for obtaining a particular result more accurately and are easier to use where a particular result needs further investigation. No one particular method of the three can be said to be better than the others. Each method has its difficulties, though not usually at the same eigenvalue. When solving a linear differential eigenvalue problem using the experience of this paper and [1] it would seem appropriate to try more than one of the methods if difficulties are encountered.

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